SPECTRA OF TWO-DIMENSIONAL TURBULENCE

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As is well known (see, e.g., [1]), the large-scale turbulence in the ocean and atmosphere can be assumed to be quasi-two-dimensional. The interest in these motions is primarily due to the fact that they possess high energies, and their role in the general circulation is substantial. In this connection we address the important problem of spectra of two-dimensional turbulence.

The dynamics of turbulence in planar flows differs substantially from that of three-dimensional flows [2, 3]. This is due to the presence of an additional integral of motion, the enstrophy (equal to half of the mean square vorticity), which exists only in the two-dimensional case. For sufficiently large Reynolds numbers the cascade process of enstrophy transfer with finite velocity ε_2 in the small-scale region becomes dominant. Dimensionality considerations in the inertia interval lead then to a turbulence spectrum of the form

$$E_k = c_2 \varepsilon_2^{2/3} k^{-3}.$$

The hypothesis of spectral enstrophy transport and the "minus three" law following from it were first formulated by Kraichnan [2].

1. We obtain this result as an exact solution of the equations of motion in the direct interaction approximation. We write the Euler equation in the Fourier representation

$$\frac{\partial v_{\mathbf{k}}^{\alpha}}{\partial t} = -\frac{i}{2} P_{\mathbf{k}}^{\alpha\beta\gamma} \int v_{\mathbf{k}_1}^{*\beta} v_{\mathbf{k}_2}^{*\gamma} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2, \tag{1.1}$$

where

$$P_{\mathbf{k}}^{\alpha\beta\gamma} = k_{\gamma}\Delta_{\mathbf{k}}^{\alpha\beta} + k_{\beta}\Delta_{\mathbf{k}}^{\alpha\gamma};\ \Delta_{\mathbf{k}}^{\alpha\beta} = \delta_{\alpha\beta} - \frac{k_{\alpha}k_{\beta}}{k^2};$$

and $\mathbf{v}_{\mathbf{k}}$ is the Fourier transform of the velocity field.

To study the statistical characteristics of Eq. (1.1) it is convenient to use Wyld's diagrammatic technique [4], which is based on two series, for the Green function $G_{\mathbf{k}\omega}^{\alpha\beta}$ and for the pair correlator of the velocity field $J_{\mathbf{k}\omega}^{\alpha\beta}$. First-order perturbation theory corresponds to the direct interaction model [5]. In the three-dimensional case this approximation gives for the spatial spectrum of the kinetic energy the asymptotic equation $J_k \sim k^{-3/2}$, which contradicts the Kolmogorov hypothesis of self-similarity. The drawback of this scheme is that it exaggerates the effect of large-scale motions on the evolution of small-scale inhomogeneities [6].

Some of the most divergent diagrams, describing pure transport, were summed up by L'vov [7]. The improved equations, as shown in [8], have an exact solution corresponding to a Kolmogorov spectrum. The subscripts of stationary helicity spectrum were found [9] within the frame of these equations. A similar procedure is used in the present paper to find spectra of two-dimensional turbulence.

The improved direct interaction equations are [8]

$$\tilde{G}_q = (\omega - \tilde{\Sigma}_q)^{-1},\ \tilde{I}_q = |\tilde{G}_q|^2 \tilde{\Phi}_q \quad (q = \mathbf{k},\ \omega),$$

$$\tilde{\Sigma}_q = \int \Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} \Gamma_{\mathbf{k}_1}^{\beta}\Big|_{\mathbf{k}\mathbf{k}_2}^{\alpha\gamma} \tilde{G}_{q_1}^{*} \tilde{I}_{q_2} [\delta(q + q_1 + q_2) - \delta(q + q_1)]\, dq_1 dq_2,$$

$$\tilde{\Phi}_q = \int \left[\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma}\right]^2 \tilde{I}_{q_1}\tilde{I}_{q_2}\left[\frac{1}{2}\delta(q + q_1 + q_2) - \delta(q + q_1)\right] dq_1 dq_2,$$

where $\tilde{G}_q^{\alpha\beta} = \tilde{G}_q\Delta_{\mathbf{k}}^{\alpha\beta};\ \tilde{J}_q^{\alpha\beta} = \tilde{I}_q\Delta_{\mathbf{k}}^{\alpha\beta};\ G_q = \langle \tilde{G}_{\mathbf{k},\omega-\mathbf{kv}}\rangle_{\mathbf{v}};\ J_q = \langle \tilde{I}_{\mathbf{k},\omega-\mathbf{kv}}\rangle_{\mathbf{v}};\ \langle\ldots\rangle_{\mathbf{v}}$ is the average over a random velocity field at an arbitrary point $\mathbf{r}$, t by means of the Wyld procedure, and

$$\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} = \left[\Delta_{\mathbf{k}}^{\alpha\alpha_1}k_{\beta_1} + \Delta_{\mathbf{k}}^{\alpha\beta_1}k_{\alpha_1}\right]\Delta_{\mathbf{k}_1}^{\alpha_1\beta}\Delta_{\mathbf{k}_2}^{\beta_1\gamma}$$

is the peak, being a homogeneous function of its arguments. In the two-dimensional case it satisfies the identities

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$$\Gamma^{\alpha}_{\mathbf{k}}\left|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2} + \Gamma^{\beta}_{\mathbf{k}_1}\right|^{\gamma\alpha}_{\mathbf{k}_2\mathbf{k}} + \Gamma^{\gamma}_{\mathbf{k}_2}\left|^{\alpha\beta}_{\mathbf{k}\mathbf{k}_1} = 0; \right. \tag{1.2}$$

$$k^2\Gamma^{\alpha}_{\mathbf{k}}\left|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2} + k_1^2\Gamma^{\beta}_{\mathbf{k}_1}\right|^{\gamma\alpha}_{\mathbf{k}_2\mathbf{k}} + k_2^2\Gamma^{\gamma}_{\mathbf{k}_2}\left|^{\alpha\beta}_{\mathbf{k}\mathbf{k}_1} = 0, \right. \tag{1.3}$$

which are consequences of the energy and enstrophy conservation laws.

We further define the following combination [10]:

$$L_q = 2i\,\mathrm{Im}\,[\widetilde{\Phi}_q \widetilde{G}_q^* + \widetilde{I}_q \textstyle\sum_q],$$

with $L_q = 0$ being equivalent to the Dyson equation for $\widetilde{I}_q$. We have

$$\begin{aligned} 0 = L_{\mathbf{k}} = \int L_{\mathbf{k}\omega} d\omega = \mathrm{Im}\int d\omega dq_1 dq_2 \delta(q + q_1 + q_2)\, \Gamma^{\alpha}_{\mathbf{k}}|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2} \\ \times \{\Gamma^{\alpha}_{\mathbf{k}}|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2}\widetilde{G}_q\widetilde{I}_{q_1}\widetilde{I}_{q_2} + \Gamma^{\gamma}_{\mathbf{k}_2}|^{\alpha\beta}_{\mathbf{k}\mathbf{k}_1}\widetilde{G}_{q_2}\widetilde{I}_q\widetilde{I}_{q_1} + \Gamma^{\beta}_{\mathbf{k}_1}|^{\gamma\alpha}_{\mathbf{k}_2\mathbf{k}}\widetilde{G}_{q_1}\widetilde{I}_{q_2}\widetilde{I}_q\}. \end{aligned} \tag{1.4}$$

In the case under consideration this equation is similar to the kinetic equation wave in the theory of weak turbulence for the nondecaying dispersion law.

First we find a solution of Eq. (1.4) describing thermodynamic equilibrium. For this we rewrite the integrand function in (1.4) in the form

$$\Gamma^{\alpha}_{\mathbf{k}}|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2}\widetilde{I}_q\widetilde{I}_{q_1}\widetilde{I}_{q_2}\,\mathrm{Im}\{\Gamma^{\alpha}_{\mathbf{k}}|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2}\widetilde{G}_q\widetilde{I}_q^{-1} + \Gamma^{\gamma}_{\mathbf{k}_2}|^{\alpha\beta}_{\mathbf{k}\mathbf{k}_1}\widetilde{G}_{q_2}\widetilde{I}_{q_2}^{-1} + \Gamma^{\beta}_{\mathbf{k}_1}|^{\gamma\alpha}_{\mathbf{k}_2\mathbf{k}}\widetilde{G}_{q_1}\widetilde{I}_{q_1}^{-1}\}\delta(q+q_1+q_2),$$

whence it is seen that due to (1.2) Eq. (1.4) admits a solution

$$\widetilde{I}_q = \frac{T}{\pi}\,\mathrm{Im}\,\widetilde{G}_q.$$

A more general solution is of the form

$$\widetilde{I}_q = \frac{1}{\pi}\,\mathrm{Im}\,G_q\,\frac{1}{a + bk^2},\quad a \text{ and } b - \text{const},$$

making (1.4) vanish due to enstrophy conservation.

Another solution of this equation, describing nonequilibrium flow distributions, is sought in the scale-invariant form

$$\widetilde{G}_q = \frac{1}{k^s}\,g\left(\frac{\omega}{k^s}\right),\quad \widetilde{I}_q = \frac{1}{k^{s+p}}\,f\left(\frac{\omega}{k^s}\right). \tag{1.5}$$

The first relation between the unknown indices is obtained from the Dyson equation

$$2s + p = 2t + d,$$

where t is the homogeneity power of the interaction parameter, and d is the dimensionality of space.

One more relation between s and p can be found by applying the factorization procedure [8, 11] to Eq. (1.4).

We perform a conformal transformation

$$k = k''\left(\frac{k}{k''}\right),\quad k_1 = k'\left(\frac{k}{k''}\right),\quad k_2 = k\left(\frac{k}{k''}\right),$$

$$\omega = \omega''\left(\frac{k}{k''}\right)^s,\quad \omega_1 = \omega'\left(\frac{k}{k''}\right)^s,\quad \omega_2 = \omega\left(\frac{k}{k''}\right)^s.$$

In this case the second term in (1.4) at the scale-invariant spectrum (1.5) transforms to the first one with a factor $(k/k_2)^x$, where $x = 2t + 2d - s - 2p$.

The third term in (1.4) is similarly transformed. As a result the integrand function is

$$\Gamma^{\alpha}_{\mathbf{k}}|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2}\widetilde{G}_q\widetilde{I}_{q_1}\widetilde{I}_{q_2}\left\{\Gamma^{\alpha}_{\mathbf{k}}|^{\beta\gamma}_{\mathbf{k}_1\mathbf{k}_2} + \left(\frac{k}{k_2}\right)^x\Gamma^{\gamma}_{\mathbf{k}_2}|^{\alpha\beta}_{\mathbf{k}\mathbf{k}_1} + \left(\frac{k}{k_1}\right)^x\Gamma^{\beta}_{\mathbf{k}_1}|^{\gamma\alpha}_{\mathbf{k}_2\mathbf{k}}\right\}.$$

The curly bracket can be made to vanish both due to energy conservation (the solution with $x = 0$) and due to enstrophy conservation (the solution with $x = -2$). As a result we have, respectively, for the spatial spectra $E_h = 2\pi k\int \widetilde{I}_{h\omega} d_\omega$

$$E_h \sim k^{-5/3}\left(x = 0,\ s = \frac{2}{3},\ p = \frac{8}{3}\right),$$

$$E_h \sim k^{-3}\ (x = -2,\ s = 0,\ p = 4).$$

2. We evaluate the energy and enstrophy flow directions from the spectrum. We write the energy balance equation

$$\frac{\partial E_k}{\partial t} = -\frac{\pi k}{2}\,\mathrm{Im}\int \Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} J_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha\beta\gamma}\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d\mathbf{k}_1 d\mathbf{k}_2,$$

where
$$J_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha\beta\gamma}\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2) = \langle v_{\mathbf{k}}^{\alpha} v_{\mathbf{k}_1}^{\beta} v_{\mathbf{k}_2}^{\gamma}\rangle.$$

This equation conserves the total energy $\int_0^\infty E_k dk$ and the total vorticity $\int_0^\infty k^2 E_k dk$. It has the form of a continuity equation, therefore the right-hand side can be represented as a divergence of corresponding flows

$$\varepsilon_i = \pi\int_0^k dk k^{i+1}\frac{1}{2}\,\mathrm{Im}\int \Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} J_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha\beta\gamma}\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d\mathbf{k}_1 d\mathbf{k}_2,$$

where $i=0$ corresponds to energy flow, and $i=2$ to enstrophy flow.

We express $J_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha\beta\gamma}$ as a series in powers of G_k and J_k. In the direct interaction approximation we have in the stationary case

$$J_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{\alpha\beta\gamma} = \int d\omega d\omega_1 d\omega_2 \delta(\omega+\omega_1+\omega_2)\Big[\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} G_q J_{q_1} J_{q_2} + \Gamma_{\mathbf{k}_2}^{\gamma}\Big|_{\mathbf{k}\mathbf{k}_1}^{\alpha\beta} G_{q_2} J_q J_{q_1} + \Gamma_{\mathbf{k}_1}^{\beta}\Big|_{\mathbf{k}_2\mathbf{k}}^{\gamma\alpha} G_{q_1} J_{q_2} J_q\Big]. \tag{2.1}$$

As a result we obtain for ε_i

$$\varepsilon_i = \pi\int_0^k k^{i+1} L_k dk. \tag{2.2}$$

For a power distribution for the spatial spectrum

$$J_k = c_i k^{-p_i} \tag{2.3}$$

L_k also has a scale-invariant form

$$L_k = D(x) k^{-d+x}.$$

We then find in the case of convergence of L_k for the distribution (2.3) the following expression for ε_i in accordance with (2.2):

$$\varepsilon_i = \pi k^{i+d} L_k/(x+i).$$

For $x=-i$ ε_i contains a singularity of type 0/0, therefore we obtain for flows at stationary spectra*

$$\varepsilon_i = \pi k^{i+d}\partial L_k/\partial x|_{x=-i}.$$

To find the derivative it is convenient to use the factorization equation for L_k in the form

$$L_k = \frac{1}{3} k^x\,\mathrm{Im}\int d\omega dq_1 dq_2 \delta(q+q_1+q_2)\left[\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} G_q J_{q_1} J_{q_2} + \Gamma_{\mathbf{k}_2}^{\gamma}\Big|_{\mathbf{k}\mathbf{k}_1}^{\alpha\beta} G_{q_2} J_q J_{q_1}\right.$$
$$\left. + \Gamma_{\mathbf{k}_1}^{\beta}\Big|_{\mathbf{k}_2\mathbf{k}}^{\gamma\alpha} G_{q_1} J_{q_2} J_q\right]\{k^{-x}\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} + k_2^{-x}\Gamma_{\mathbf{k}_2}^{\gamma}\Big|_{\mathbf{k}\mathbf{k}_1}^{\alpha\beta} + k_1^{-x}\Gamma_{\mathbf{k}_1}^{\beta}\Big|_{\mathbf{k}_2\mathbf{k}}^{\gamma\alpha}\}.$$

Hence we have for ε_i

$$\varepsilon_i = -\frac{\pi}{3} c_i^{3/2} k^d \int d\mathbf{k}_1 d\mathbf{k}_2 \delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,\theta_{kk_1k_2}(kk_1k_2)^{-p_i}$$
$$\times\left[k^{p_i}\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} + k_2^{p_i}\Gamma_{\mathbf{k}_2}^{\gamma}\Big|_{\mathbf{k}\mathbf{k}_1}^{\alpha\beta} + k_1^{p_i}\Gamma_{\mathbf{k}_1}^{\beta}\Big|_{\mathbf{k}_2\mathbf{k}}^{\gamma\alpha}\right]\left\{k^i \ln k\,\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma} + k_2^i \ln k_2\,\Gamma_{\mathbf{k}_2}^{\gamma}\Big|_{\mathbf{k}\mathbf{k}_1}^{\alpha\beta} + k_1^i \ln k_1\,\Gamma_{\mathbf{k}_1}^{\beta}\Big|_{\mathbf{k}_2\mathbf{k}}^{\gamma\alpha}\right\}, \tag{2.4}$$

where $\theta_{kk_1k_2} = \int_0^\infty g(k^s t)\,f(k_1^s t)\,f(k_2^s t)\,dt$ is a positive-definite function (see, e.g., [3]), which for corresponding approximations can be performed symmetrically in permutations of their arguments [2, 13, 14]. It is seen from (2.4) that $c_i \sim \varepsilon_i^{2/3}$, which, naturally, also follows from dimensionality considerations.

In the two-dimensional case it follows from Eqs. (1.2), (1.3) that

$$\Gamma_{\mathbf{k}_2}^{\gamma}\Big|_{\mathbf{k}\mathbf{k}_1}^{\alpha\beta} = \frac{k^2-k_1^2}{k_1^2-k_2^2}\,\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma},$$
$$\Gamma_{\mathbf{k}_1}^{\beta}\Big|_{\mathbf{k}_2\mathbf{k}}^{\gamma\alpha} = \frac{k_2^2-k^2}{k_1^2-k_2^2}\,\Gamma_{\mathbf{k}}^{\alpha}\Big|_{\mathbf{k}_1\mathbf{k}_2}^{\beta\gamma}. \tag{2.5}$$

*A similar relation for flows holds in the theory of weak turbulence [12].

Substituting (2.5) into (2.4) and introducing new variables by $k_1 = kv$, $k_2 = ku$, we finally obtain for ε_i

$$\varepsilon_i = \frac{\pi}{3} c_i^{3/2} \iint_\Delta \frac{du\,dv\,\theta_{1vu} \left[\Gamma_1^\alpha \left|_{vu}^{\beta\gamma}\right.\right]^2 (uv)^{-p_i+1}}{(u^2 - v^2) \sqrt{[(u+v)^2 - 1][1 - (u-v)^2]}} \times \left[u^2 - v^2 + (v^2 - 1) u^{p_i} - (u^2 - 1) v^{p_i}\right] \{(v^2 - 1) u^i \ln u - (u^2 - 1) v^i \ln v\}. \tag{2.6}$$

The region of integration Δ is determined by the triangular inequaltiy with sides u, v, and 1: $|u-v| \leq 1 \leq u+v$. The flow directions are given by the sign of the integral (2.6); for sign $\varepsilon_i > 0$ the flow is directed to the range of large wave numbers, and for the opposite inequality – toward small wave numbers.

It follows from the analysis (see Appendix) that the integrand function in (2.6) is sign-definite, with

$$\operatorname{sign} \varepsilon_0 = -\operatorname{sign} p_0(p_0 - 2), \quad \operatorname{sign} \varepsilon_2 = \operatorname{sign} p_2(p_2 - 2).$$

For the nonequilibrium solutions $p_0 = 8/3$ and $p_2 = 4$ we have in this case

$$\operatorname{sign} \varepsilon_0 < 0, \quad \operatorname{sign} \varepsilon_2 > 0.$$

The result obtained can be found in [15, 16] by other considerations.

3. For the results obtained to have physical sense it is necessary to prove the locality of turbulence. The latter physically implies that vortex interaction with scales of the same order are much stronger than vortex interactions of different scales. Formally the locality property requires that the integrals in (1.4) converge. Elementary analysis shows that the integrals in (1.4) converge at the upper limit for both spectra.

Consider the convergence in the region $q_1 \ll q$ ($q_2 \sim q$). In this case the most dangerous terms with which most of the divergence in (1.4) is related are the terms proportional to J_{q_1}:

$$\operatorname{Im} \int dq_1 \Gamma_k^\alpha \left|_{k_1 - (k+k_1)}^{\beta\gamma}\right. \left\{\Gamma_k^\alpha \left|_{k_1 - (k+k_1)}^{\beta\gamma}\right. G_q J_{-(q+q_1)} + \Gamma_{-(k+k_1)}^\gamma \left|_{kk_1}^{\alpha\beta}\right. G_{-(q+q_1)} J_q\right\} J_{q_1}.$$

For small q_1 with account of $\Gamma_k^\alpha \left|_{k_1 k_2}^{\beta\gamma}\right. \simeq \Gamma_k^\alpha \left|_{k_1 k}^{\beta\gamma}\right.$ and the use of identity (1.2) and the property $G_q = -G_{-q}^*$ these terms are grouped into the expression

$$\int_0^{2k} \frac{dk_1}{\sqrt{4k^2 - k_1^2}} \Gamma_{k_1}^\gamma \left|_{kk}^{\alpha\beta}\right. \Gamma_k^\alpha \left|_{k_1 k}^{\beta\gamma}\right. J_{k_1}. \tag{3.1}$$

The integration is carried over the region where the radicand is positive.

Since the spectrum is $\Gamma_{k_1}^\gamma \left|_{kk}^{\alpha\beta}\right. \Gamma_k^\alpha \left|_{k_1 k}^{\beta\gamma}\right. \equiv 0$,* this guarantees convergence of the integral (3.1) and implies locality of the spectra obtained.

4. Since the enstrophy flow is directed toward large wave numbers, when the wave vector approaches the internal Kolmogorov scale $\eta_2 = (\nu/\varepsilon_2^{1/3})^{1/2}$ (ν is the molecular viscosity coefficient) the vorticity spectrum must vanish. We find the correction to the vorticity spectrum due to the vanishing effect in the region $k\eta_2 < 1$, where direct account of viscosity in the hydrodynamic equations cannot be taken.

For this purpose we linearize Eq. (1.4) over the vorticity spectrum. The perturbations are represented in the form

$$\delta J_{k\omega} = k^{-(s_2 + p_2 + \alpha)} f_2\left(\frac{\omega}{k^{s_2}}\right), \quad \delta G_{k\omega} = k^{-(s_2 + \alpha)} g_2\left(\frac{\omega}{k^{s_2}}\right),$$

where α is an unknown power determined by the vanishing effect.

After integration over k the linearized equation $\delta L_q = 0$ acquires the following form for the correction:

$$\int \delta L_{k\omega} d\mathbf{k} = \operatorname{Im} \int \Gamma_k^\alpha \left|_{k_1 k_2}^{\beta\gamma}\right. \delta J_{kk_1k_2}^{\alpha\beta\gamma} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2 = 0,$$

where $\delta J_{kk_1k_2}^{\alpha\beta\gamma}$ is the linearized expression for $J_{kk_1k_2}^{\alpha\beta\gamma}$ of (2.1). After a conformal transformation over frequency we reach the equation

$$\operatorname{Im} \int d\mathbf{k}\, dq_1\, dq_2\, \delta(q + q_1 + q_2) \left[\Gamma_k^\alpha \left|_{k_1 k_2}^{\beta\gamma}\right. + \left(\frac{\omega}{\omega_1}\right)^y \Gamma_{k_1}^\beta \left|_{k_2 k}^{\gamma\alpha}\right. + \left(\frac{\omega}{\omega_2}\right)^y \Gamma_{k_2}^\gamma \left|_{kk_1}^{\alpha\beta}\right.\right] \delta J_{qq_1q_2}^{\alpha\beta\gamma} = 0,$$

where $y = (1/s_2)(2 + \alpha)$, $J_{kk_1k_2}^{\alpha\beta\gamma} = \int \delta(\omega + \omega_1 + \omega_2)\, d\omega\, d\omega_1\, d\omega_2\, J_{qq_1q_2}^{\alpha\beta\gamma}$.

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*This upper limit singularity was noted in [9].

By identity (1.2) this expression vanishes at $y=0$, whence $\alpha=-2$. Thus, $\delta J_k \sim k^2 J_k$. At the viscous boundary $\delta J_k \sim J_k$, and the power spectrum must be cut off [7]. We then have for $k\eta_2<1$

$$E_k = c_2 \varepsilon_2^{2/3} k^{-3}[1 - c(k\eta_2)^2],$$

where c is a positive dimensionless coefficient of order unity.

We note that the viscous correction to frequency νk^2 is $(k\eta_2)^2$ times smaller than the Kolmogorov frequency $\omega_k \sim \varepsilon_2^{1/3} k^0$, therefore direct account of viscosity leads to a correction to J_k of the same order:

$$\delta J_k/J_k \sim c_3(k\eta_2)^2,\ c_3 \sim 1.$$

Further turning attention to the fact that if the viscosity is nonvanishing an interval of viscous dissipation is generated, an energy flow $\tilde{\varepsilon}_0$ appears in this case in the region of large wave numbers. For low viscosity this flow is small:

$$\tilde{\varepsilon}_0 \sim \eta_2^2 \varepsilon_2$$

and the statistical regime in this region is mostly determined by the enstrophy flow.

5. The single-flow spectra obtained above must be realized in two inertial intervals, while the spectrum $E_k \sim \varepsilon_0^{2/3} k^{-5/3}$ is realized for $k \ll k_0$, where $k_0^{-1}=L$ is the scale of excited turbulence, and $E_k \sim \varepsilon_2^{2/3} k^{-3}$ for $k \gg k_0$.

Two-flow distributions are realized under real conditions: A correction with constant energy appears on the background of a Kolmogorov spectrum with constant vorticity, and vice versa.* The two-flow distribution can be expressed in terms of the dimensionless function F depending on the flow ratio.

In this case the self-similar solution is

$$E_k = \varepsilon_2^{2/3} k^{-3} F\left(\frac{k^2 \varepsilon_0}{\varepsilon_2}\right).$$

The explanation of specific structures of two-flow distributions requires separate treatment.

APPENDIX

Due to the symmetry of the integrand function in (2.6) and the integration region in $u \rightleftharpoons v$ it is sufficient to consider the case $u>v$ to determine the sign of the integral. In this case it is necessary to consider three regions: 1) $0<v<1$, $u>1$; 2) $v>1$, $u>1$; 3) $0<v>1$, $0<u<1$. The sign of the integral is determined by the sign of the product

$$[u^2 - v^2 + (v^2-1)u^{p_i} - (u^2-1)v^{p_i}]\{(v^2-1)u^i \ln u - (u^2-1)v^i \ln v\}.$$

We determine the signs of the factors in the regions under consideration

$$K(u, v, p_i) \equiv [u^2 - v^2 + (v^2-1)u^{p_i} - (u^2-1)v^{p_i}],$$

$$K(u, 1, p_i) = 0,\ \frac{\partial K}{\partial v} = 2v\left(u^{p_i} - 1\right) - p_i v^{p_i - 1}(u^2-1) \equiv F(u, v, p_i),$$

$$F(1, v, p_i) = 0,\ \frac{\partial F}{\partial u} = 2p_i uv\left(u^{p_i-2} - v^{p_i-2}\right).$$

For $u>v$ $\operatorname{sign}(\partial F/\partial u) = \operatorname{sign} p_i(p_i-2)$.

Then for $0<v<1$, $u>1$ sign $F(u, v, p_i) \equiv$ sign $(\partial K/\partial v)$ = sign $p_i(p_i-2)$ and sign $K(u, v, p_i) = -$sign $p_i(p_i-2)$.

For $v>1$, $u>1$ sign $F(u, v, p_i) \equiv$ sign $(\partial K/\partial v)$ = sign $p_i(p_i-2)$ and sign $K(u, v, p_i)$ = sign $p_i(p_i-2)$.

For $0<v<1$, $0<u<1$ sign $(\partial K/\partial v) = -$sign $p_i(p_i-2)$ and sign $K(u, v, p_i) = -$sign $(\partial K/\partial v)$ = sign $p_i(p_i-2)$.

Consider the sign of the function

$$\Phi_i(u, v) \equiv (v^2-1)u^i \ln u - (u^2-1)v^i \ln v \equiv (v^2-1)(u^2-1)[\varphi_i(u) - \varphi_i(v)],$$

where $\varphi_i(x) = \frac{x^i \ln x}{x^2-1}$, $i = 0, 2$.

a) $\varphi_0(x) = \frac{\ln x}{x^2-1}$; since $\varphi_0'(x)<0$, $\varphi_0(u) - \varphi_0(v) < 0$ for $u>v$. We then have in the regions $u>1$, $v>1$ and $0<v<1$, $0<u<1$ $\Phi_0(u, v)<0$, and for $0<v<1$, $u>1$ $\Phi_0(u, v)>0$;

*This is indicated by numerical calculations (see, e.g., [17]).

b) $\varphi_2(x) = \frac{x^2 \ln x}{x^2 - 1}$; since $\varphi_2'(x) > 0$ it follows that $\varphi_2(u) - \varphi_2(v) > 0$ for $u > v$. In this case we have in the regions where $u > 1$, $v > 1$ and $0 < v < 1$, $0 < u < 1$, $\Phi_2(u, v) > 0$ and $\Phi_2(u, v) < 0$ for $0 < v < 1$, $u > 1$. Thus,

$$\text{sign}\ \varepsilon_0 = \text{sign}\ K \cdot \text{sign}\ \Phi_0 = -\text{sign} p_0(p_0 - 2),\ \text{sign}\ \varepsilon_2 = \text{sign}\ K \cdot \text{sign}\ \Phi_2 = \text{sign}\ p_2(p_2 - 2).$$

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